Tournament Solutions

and their Applications to Multiagent Decision-Making

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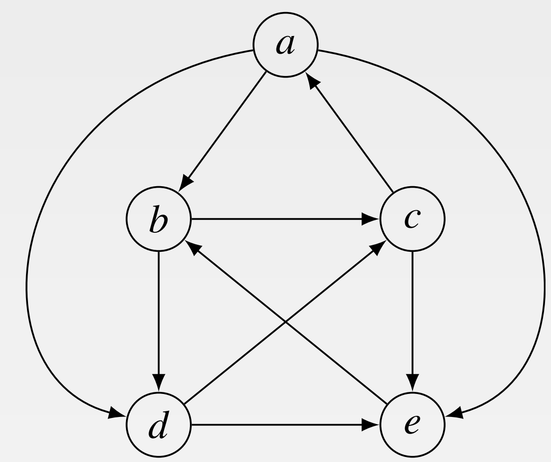
PREFERENCE AGGREGATION IN MULTIAGENT SYSTEMS

Multiagent Decision-Making

- Collective decision-making
  - *Social choice theory*
  - How to aggregate the possibly conflicting preferences of multiple agents?
- Adversarial decision-making
  - *Theory of zero-sum games*
  - Which strategy should be pursued when interacting with other agents?
- Coalitional decision-making
  - *Cooperative game theory*
  - Which coalitions of agents will form and how should they divide the proceeds of cooperations?

The Trouble with Tournaments

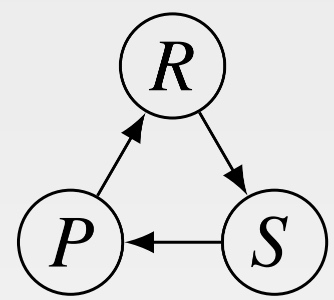


- Problem: Find “best” elements according to some binary dominance relation
    - Set of alternatives is finite
    - Dominance relation is asymmetric and complete
    - Maximal elements need not exist (“Condorcet paradox” in *social choice theory*)
    - According to *game theorists* von Neumann & Morgenstern (1944), cyclical dominations are “one of the most characteristic difficulties which a theory of these phenomena must face.”
- Numerous applications
    - social choice theory, cooperative and non-cooperative game theory
        - also multi-criteria decision analysis, sports tournaments, psychometrics, biology, argumentation theory, webpage and journal ranking, etc.

Social Choice & Game Theory

$\succ_1$	$\succ_2$	$\succ_3$
S	R	P
P	S	R
R	P	S

Condorcet Paradox



$$\begin{array}{c c} & \begin{array}{ccc} R & P & S \end{array} \\ \begin{array}{c} R \\ P \\ S \end{array} & \begin{pmatrix} 0 & -1 & +1 \\ +1 & 0 & -1 \\ -1 & +1 & 0 \end{pmatrix} \end{array}$$

Rock Paper Scissors

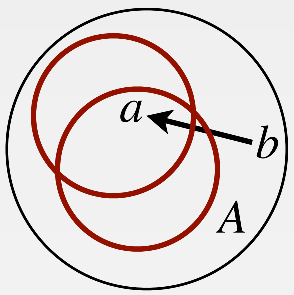
- Theorem (McGarvey, 1953): Any asymmetric relation can be induced by majority rule.
    - When the number of voters is odd and individual preferences are strict, then any relation induced by majority rule is asymmetric and complete.
- Tournament games
    - Subclass of symmetric two-player zero-sum games
        - players may only win, lose, or draw
        - game ends in draw iff both players play the same action

Terminology & Notation

- Finite set of alternatives A
- Asymmetric and complete dominance relation $\succ \subseteq A \times A$
- Tournament $T = (A, \succ)$
- $\mathcal{T}(A)$: Set of all tournaments on A
- Tournament solution S associates with each tournament T a non-empty subset of alternatives (the "winners" of T according to S)
  $S : \mathcal{T}(A) \to \mathcal{P}(A) \setminus \emptyset$
  - stable with respect to tournament isomorphisms: $S \circ \pi = \pi \circ S$
  - selects maximal element whenever one exists: $\max_{\succ}(T) \subseteq S(T)$
- $S(B) = S(T|_B) = S((B, \succ|_B))$ for $B \subseteq A$
- Dominion $D_{(A,\succ)}(a) = \{ b \in A \mid a \succ b \}$
- Dominators $\overline{D}_{(A,\succ)}(a) = \{ b \in A \mid b \succ a \}$

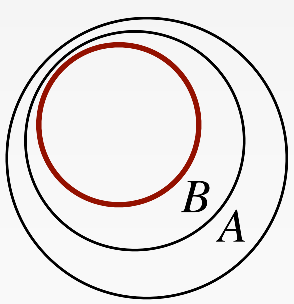
Desirable Properties

- Monotonicity (MON)
  - An alternative remains in the choice set when its dominion is extended.



$$a \in S(T) \Rightarrow a \in S(T') \quad \forall T = (A, \succ), T' = (A, \succ'), a \in A:$$
$$T|_{A\setminus\{a\}} = T'|_{A\setminus\{a\}} \wedge D_T(a) \subseteq D_{T'}(a)$$

- Strong Superset Property (SSP)
  - The choice set is invariant under removal of unchosen alternatives.



$$S(B) = S(A) \quad \forall T = (A, \succ), B \subseteq A:$$
$$S(A) \subseteq B$$

- **Further properties:** idempotency (IDE), weak superset property (WSP), independence of unchosen alternatives (IUA), composition-consistency (COM), irregularity (IRR)

Tournament Solution Hierarchy

(B., 2008)

via qualified subsets	*via stable sets*

Banks set (Banks, 1985)

Minimal covering set (Dutta, 1988)

Tournament equilibrium set (Schwartz, 1990)

Copeland set (Zermelo, 1929; Copeland, 1951)

Uncovered set (Fishburn, 1977; Miller, 1980)

Minimal extending set

Condorcet non-losers (Condorcet, 1785)

Bipartisan set (Laffond et al., 1993)

Top cycle (Good, 1971; Smith, 1973)

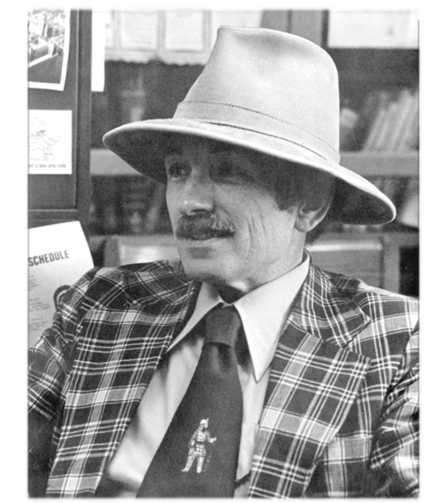
Markov set (Daniels, 1969)

Slater set (Slater, 1961)

- Infinite hierarchy of tournament solutions
  - improve understanding of tournament solutions and their relationships
  - unify proofs of properties and inclusions
  - define new tournament solutions, e.g., minimal extending set

The Top Cycle

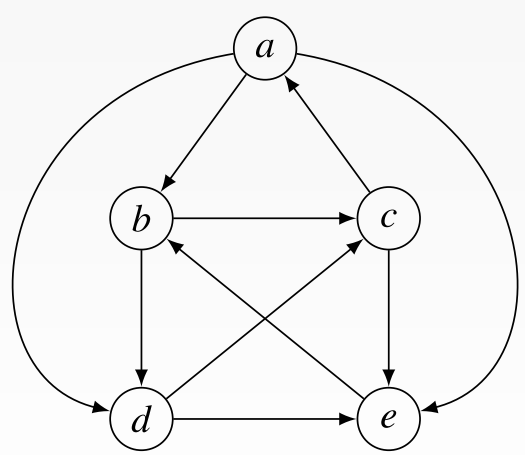
(Good, 1971; Smith, 1973)


John I. Good

- A dominating set is a set of alternatives such that every alternative in the set dominates every outside alternative.
  - The set of all dominating sets is totally ordered by set inclusion.
  - Every tournament contains a unique minimal dominating set.
- The minimal dominating set is called the top cycle (TC).
  - also known as GETCHA or Smith set
- Theorem (Bordes, 1976): The top cycle is the smallest tournament solution satisfying β^+.
  - It also satisfies WSP, SSP, MON, IUA.
- How can we efficiently compute the top cycle?

TC (linear-time algorithm)

- Algorithm for computing TC_a, the minimal dominating set containing a given alternative a
  - Initialize working set B with {a} and then iteratively add all alternatives that dominate an alternative in B until no more such alternatives can be found.
  - Computing TC_a for every alternative a and then choosing the smallest set yields an $O(n^3)$ algorithm.
- Alternatives with maximal degree (the Copeland winners) are always contained in TC (and linear-time computable).
  - **procedure** $TC(A, \succ)$
    $B \leftarrow C \leftarrow CO(A, \succ)$
    loop
    $C \leftarrow \bigcup_{a \in C} \overline{D}_{A \setminus B}(a)$
    if $C = \emptyset$ **then return** B **end if**
    $B \leftarrow B \cup C$
    end loop



More on the Top Cycle

- Theorem (Deb, 1977): The top cycle consists precisely of the maximal elements of the asymmetric part of the transitive closure of the dominance relation.
  - Alternative linear-time algorithm using Kosaraju's or Tarjan's algorithm for finding strongly connected components
- There is a first-order expression for membership in TC (B., Fischer, & Harrenstein; 2009):

$$TC(x) \leftrightarrow \forall y \forall z (\forall v (z \geq^3 v \rightarrow z \geq^2 v) \wedge z \geq^2 x \rightarrow z \geq^2 y)$$

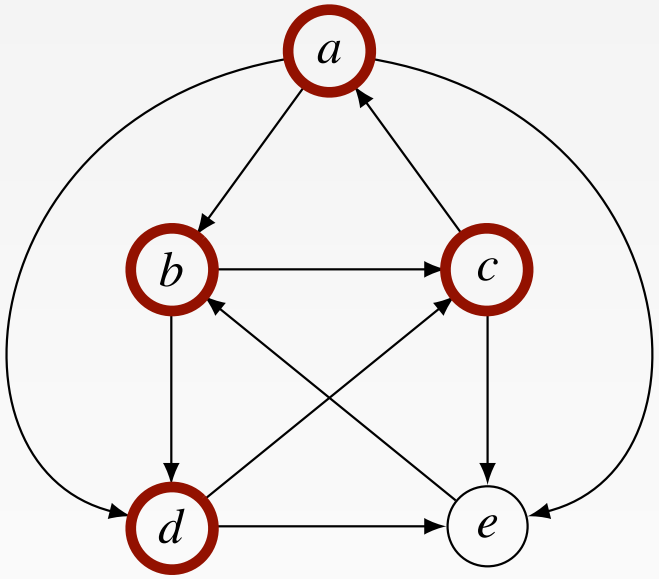
  - Computing TC is in AC^0
- The top cycle is very large.
  - In fact, it is so large that it may contain Pareto-dominated alternatives when used as a social choice function.

The Uncovered Set

(Fishburn, 1977; Miller, 1980)


Nicholas Miller

- Covering relation: a covers b if $D(b) \subset D(a)$.
  - The covering relation is a transitive subrelation of the dominance relation.
- The uncovered set (UC) consists of all uncovered alternatives.
  - UC contains the maximal element of inclusion-maximal subsets that admit a maximal element.
- Example
  - UC = {a,b,c,d}



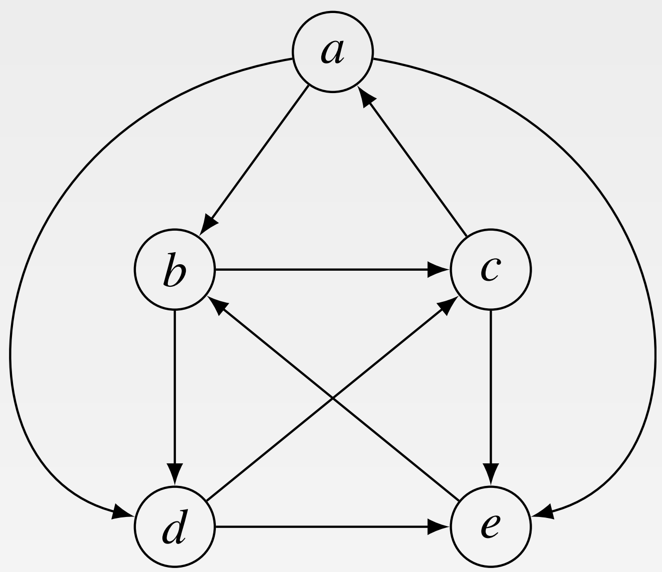
- Theorem (Moulin, 1986): The uncovered set is the smallest tournament solution satisfying γ.
  - It also satisfies WSP, MON, and COM and is contained in TC.

UC algorithm

- Straightforward n^3 algorithm
- Equivalent characterization of UC
  - UC consists precisely of those alternatives that reach every other alternative on a domination path of length at most two. (Shepsle & Weingast, 1984).
- Algorithm via matrix multiplication
  - Fastest known matrix multiplication algorithm (Coppersmith & Winograd, 1990): $O(n^{2.38})$
  - Matrix multiplication is believed to be feasible in linear time ($O(n^2)$).

procedure $UC(A, \succ)$
  for all $i, j \in A$ **do**
    if $i \succ j \vee i = j$ **then** $m_{ij} \leftarrow 1$
    else $m_{ij} \leftarrow 0$ **end if**
  end for
  $M \leftarrow (m_{ij})_{i,j \in A}$
  $U \leftarrow (u_{ij})_{i,j \in A} \leftarrow M^2 + M$
  $B \leftarrow \{i \in A \mid \forall j \in A : u_{ij} \neq 0\}$
  return B

UC algorithm (example)



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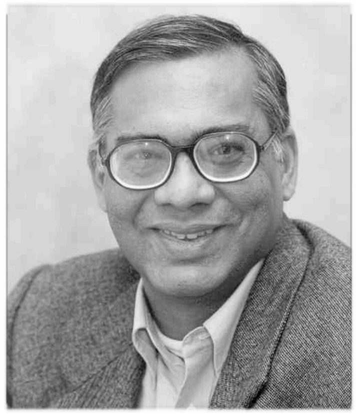
  return B

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}^2 + \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

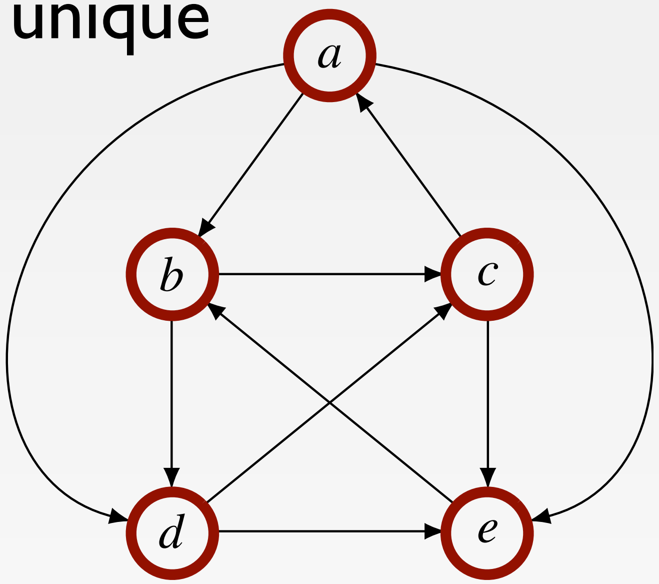
The Minimal Covering Set

(Dutta, 1988)



Bhaskar Dutta

- A covering set is a set of alternatives B such that a∉UC(B∪{a}) for all alternatives a∉B.
  - Theorem (Dutta, 1988): Every tournament contains a unique minimal covering set (MC).



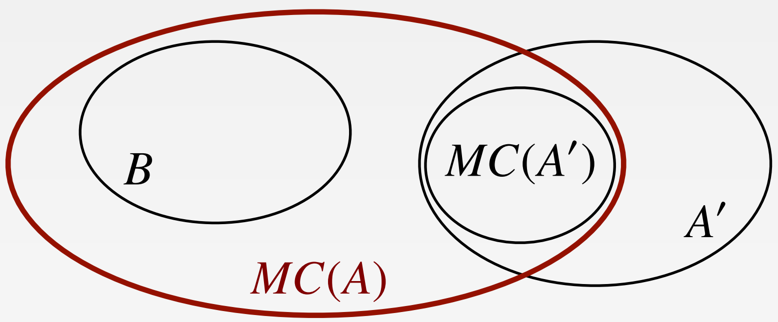
- Example
  - Covering sets: {a,b,c,d,e}, {a,b,c,d}, and {a,b,c}
  - MC = {a,b,c}
- Theorem (Dutta, 1988): The minimal covering set is the smallest tournament solution satisfying SSP and γ*.
  - It also satisfies WSP, MON, IUA, and COM and is contained in UC.
  - MC is equivalent to a game-theoretic concept proposed by Shapley in 1953 (Duggan & Le Breton, 1996)

MC (complexity)

- No obvious reason why computing MC should be in NP
  - Verifying whether a set is a covering set is easy, verifying minimality is not.
  - Checking whether a set is MC and checking whether an alternative is contained in MC is in coNP.
    - A covering set is *not* minimal if there exists a proper covering subset.
- Straightforward iterative algorithms do not work
  - start with entire set and remove alternatives
    - there may be no covering sets in between entire set and MC
  - start with singleton and add alternatives
    - unclear which of the alternatives that are not covered by the current working set should be included

MC (algorithm)

- Three insights needed for polynomial-time algorithm
  - Lemma: If $B \subseteq MC(A)$ and $A' = \bigcup_{a \in A \setminus B} (UC(B \cup \{a\}) \cap \{a\})$ then $MC(A') \subseteq MC(A)$.



    - For every proper subset of MC, the lemma tells us how to find another disjoint and non-empty subset of MC.
  - Lemma (Laffond, Laslier, & Le Breton; 1993): Every tournament game contains a unique Nash equilibrium, the support of which (the so-called bipartisan set BP) is contained in MC.
  - The bipartisan set can be computed via linear programming.

MC (algorithm, ctd.)

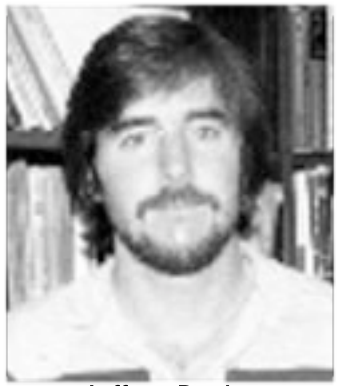
- Theorem (B. and Fischer, 2008): The minimal covering set can be computed in polynomial time.

procedure $MC(A, \succ)$
  $B \leftarrow BP(A, \succ)$
  loop
    $A' \leftarrow \bigcup_{a \in A \setminus B} (UC(B \cup \{a\}) \cap \{a\})$
    if $A' = \emptyset$ **then return** B **end if**
    $B \leftarrow B \cup BP(A', \succ)$
  end loop

procedure $BP(A, \succ)$
  for all $i, j \in A$ **do**
    if $i \succ j$ **then** $m_{ij} \leftarrow 1$
    else if $j \succ i$ **then** $m_{ij} \leftarrow -1$
    else $m_{ij} \leftarrow 0$ **end if**
  end for

$$s \in \{s \in \mathbb{R}^n \mid \begin{array}{ll} \sum_{j \in A} s_j \cdot m_{ij} \leq 0 & \forall i \in A \\ \sum_{j \in A} s_j = 1 & \\ s_j \geq 0 & \forall j \in A\} \end{array}$$

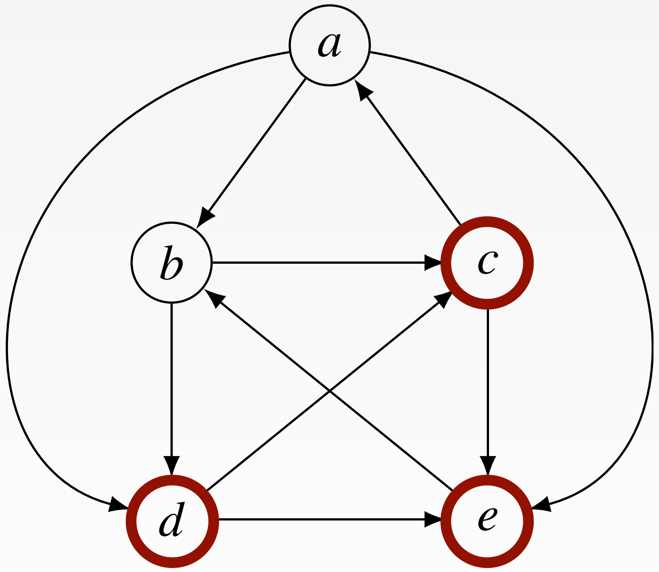
  $B \leftarrow \{a \in A \mid s_a > 0\}$
  return B


Jeffrey Banks

Banks set

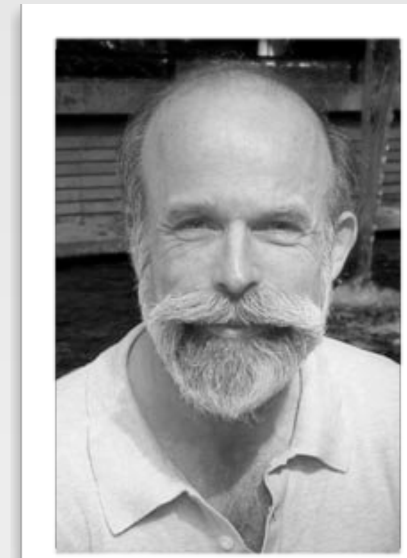
(Banks, 1985)

- The Banks set (BA) consists of the maximal elements of maximal transitive subsets.
- Theorem (B., 2008): The Banks set is the smallest tournament solution satisfying strong retentiveness.
  - It also satisfies WSP, MON, IRR, COM, and is contained in UC.
- Random alternatives in BA can be found efficiently.
  - BA = {a,b,c,d}
- Theorem (Woeginger, 2003): Deciding whether a given alternative is contained in BA is NP-complete.



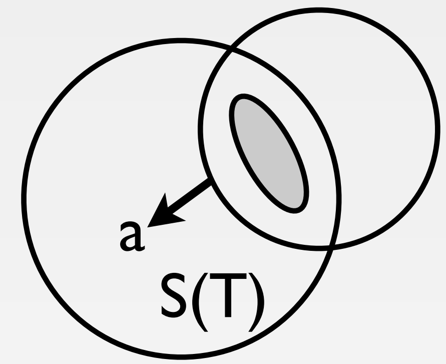
Tournament Equilibrium Set

(Schwartz, 1990)

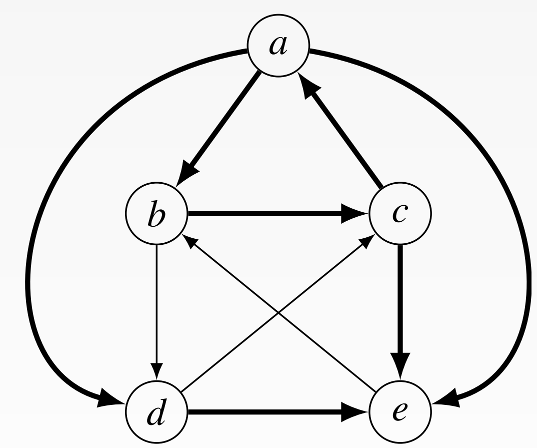


Thomas Schwartz

- A tournament solution S is retentive if $S(\overline{D}(a)) \subseteq S(T)$ for all $a \in S(T)$ and all tournaments T.
  - Idea: No alternative in the choice set should be "properly" dominated by an outside alternative.

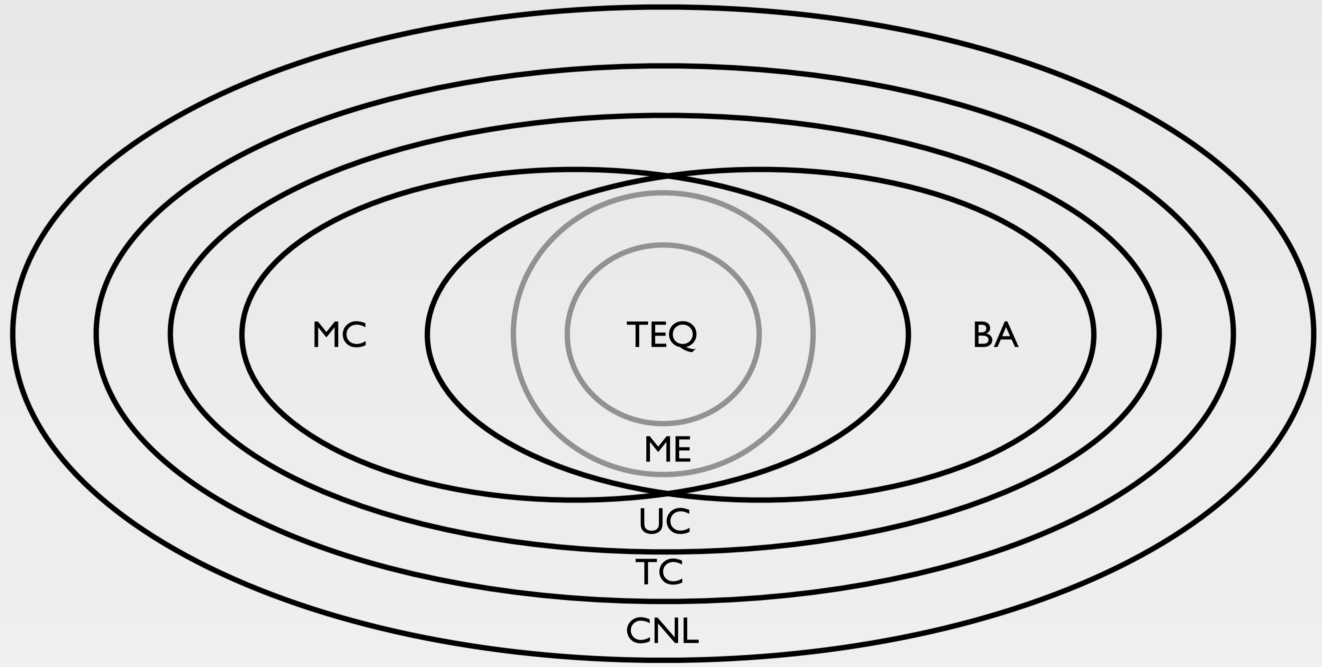


- TEQ is the smallest tournament solution satisfying retentiveness.
  - Characterization relies on Schwartz's conjecture.
  - TEQ satisfies IRR and COM and is contained in BA.
- Example: TEQ = {a,b,c}



The Mystery of TEQ

- Theorem (Laffond et al., 1993; Houy, 2009): The following statements are equivalent:
  - Schwartz's conjecture
  - TEQ satisfies WSP.
  - TEQ satisfies SSP.
  - TEQ satisfies MON.
  - TEQ satisfies IUA.
- Furthermore, these statements imply that TEQ is contained in MC.
- All or nothing: Either TEQ is a most appealing tournament solution or it is severely flawed.
- Theorem (B., Fischer, Harrenstein, Mair; 2010): Deciding whether an alternative is contained in TEQ is NP-hard.
  - The best known upper bound is PSPACE!



			MON	WSP	SSP	IDE	IUA	COM	IRR	EFFICIENTLY COMPUTABLE
$S_{\mathcal{M}_2}$	CNL	(Condorcet, 1785)	✓	✓	-	-	✓	-	-	✓
$\vdots$			✓	✓	-	-	-	-	-	✓
$S_{\mathcal{M}}$	UC	(Fishburn, 1977; Miller, 1980)	✓	✓	-	-	-	✓	-	✓
$S_{\mathcal{M}^*}$	BA	(Banks, 1985)	✓	✓	-	-	-	✓	✓	NP-hard
$\widehat{S}_{\mathcal{M}_2}$	TC	(Good, 1971; Smith, 1973)	✓	✓	✓	✓	✓	-	-	✓
$\vdots$			✓	✓	✓	✓	✓	-	-	✓
$\widehat{S}_{\mathcal{M}}$	MC	(Dutta, 1988)	✓	✓	✓	✓	✓	✓	-	✓
$\widehat{S}_{\mathcal{M}^*}$	ME	(Brandt, 2008)	✓	✓	✓	✓	✓	✓	✓	NP-hard
$\widehat{TEQ}$	TEQ	(Schwartz, 1990)	✓	✓	✓	✓	✓	✓	✓	NP-hard